UNIFIED MODEL DOCUMENTATION PAPER NO 13

DERIVATION AND CALCULATION OF UNIFIED MODEL POTENTIAL VORTICITY AND THE CALCULATION OF POTENTIAL TEMPERATURE ON P V SURFACES

by

S R Anderson and I Roulstone

Version 1

9th February 1993

Model Version 3.1

Numerical Weather Prediction
Meteorological Office
London Road
BRACKNELL
Berkshire
RG12 2SZ
United Kingdom

Modification Record		
Document version	**Author**	**Description....................**

INTRODUCTION

In this document we shall refer to two sets of equations used in Numerical Weather Prediction models. The *Hydrostatic Primitive Equations* (which are used in most NWP models) take the following form in height co-ordinates

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{a\cos\varphi}\right) v \sin\varphi + \frac{1}{\rho\, a\cos\varphi}\frac{\partial p}{\partial \lambda} = F_\lambda \qquad (1)$$

$$\frac{Dv}{Dt} + \left(2\Omega + \frac{u}{a\cos\varphi}\right) u \sin\varphi + \frac{1}{\rho\, a}\frac{\partial p}{\partial \varphi} = F_\varphi \qquad (2)$$

$$g + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \qquad (3)$$

$$\frac{D\theta}{Dt} = F_\theta \qquad (4)$$

$$\frac{D\rho}{Dt} + \rho\, \underline{\nabla}.\underline{u} = 0 \qquad (5)$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{a\cos\varphi}\frac{\partial}{\partial \lambda} + \frac{v}{a}\frac{\partial}{\partial \varphi} + w\frac{\partial}{\partial z},$$

(λ, φ, r) are the spherical polar co-ordinates denoting east longitude and latitude in radians, and distance from the centre of the Earth,

r = a + z

a is the mean radius of the Earth (6371229m),

z is height above mean sea level,

(u,v,w) are the (λ, φ, r) components of the Eulerian flow velocity vector,

t is time,

Ω is the mean angular velocity of the Earth
(7.292116×10^{-5} radians s^{-1}),

g is the acceleration due to gravity and centripetal effects at the Earth's surface (average value 9.80665 ms^{-2}),
ρ is the density,

and F_λ , F_φ and F_θ are source terms.

Omitted terms include those in $2\Omega\cos\varphi$, a number of metric terms and the vertical acceleration. The shallow atmosphere approximation has also been made where r has been replaced by a, the Earth's mean radius, except in the derivative terms $\frac{\partial}{\partial r}$ (which are retained as $\frac{\partial}{\partial z}$).

For frictionless, adiabatic flow, F_λ , F_φ and F_θ are zero, and the potential vorticity q is conserved following a fluid parcel:

$$\frac{Dq}{Dt} = 0 \qquad (6)$$

where

$$q \equiv \frac{1}{\rho}\left[-\frac{1}{a\cos\varphi}\frac{\partial v}{\partial z}\frac{\partial\theta}{\partial\lambda} + \frac{1}{a}\frac{\partial u}{\partial z}\frac{\partial\theta}{\partial\varphi} + \left(2\Omega\sin\varphi + \frac{1}{a\cos\varphi}\left(\frac{\partial v}{\partial\lambda} - \frac{\partial(u\cos\varphi)}{\partial\varphi}\right)\right)\frac{\partial\theta}{\partial\varphi}\right] \qquad (7)$$

In isentropic co-ordinates, (7) becomes

$$q \equiv -g\frac{\partial\theta}{\partial p}\left[2\Omega\sin\varphi + \frac{1}{a\cos\varphi}\left(\frac{\partial v}{\partial\lambda} - \frac{\partial(u\cos\varphi)}{\partial\varphi}\right)\right] \qquad (8)$$

where we have used the chain rule

$$\left.\frac{\partial\cdot}{\partial c}\right|_z = \left.\frac{\partial\cdot}{\partial c}\right|_\theta - \left.\frac{\partial\cdot}{\partial z}\frac{\partial z}{\partial c}\right|_\theta ,$$

where *c* may be λ , φ ,or *t*. In (8) we have used the hydrostatic equation (3). Also note that the partial derivatives with respect to the angular variables are calculated at constant θ .

The *Quasi-Hydrostatic Equations* used in the Unified Model have a complete representation of the Coriolis terms and they do not have any metric terms omitted. Another new feature is the use of a *pseudo-radius* $r_s(p)$, in place of the height *r*. The shallow atmosphere approximation is not applied. The pseudo-radius is defined as
where R is the gas constant for dry air, and $T_s(p)$ is to be interpreted as a typical profile

$$r_s(p) = a + \int_p^{p_o} \frac{RT_s(p')}{gp'} dp'. \tag{9}$$

representing the horizontally averaged, hydrostatically balanced state of the atmosphere. In the Unified Model, T_s is calculated based on an ICAO standard atmosphere (see UMDP No.10, appendix 2 for further details).

Differentiating equation (9) with respect to t gives

$$\frac{Dr_s}{Dt} = -\frac{RT_s^{(p)}\omega}{gp} \equiv \tilde{w} \tag{10}$$

where $\omega = \frac{Dp}{Dt}$. This is a key element in the dynamical consistency of the new model (see White and Bromley). The material derivative in pressure co-ordinates takes the form

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{r_s\cos\varphi}\frac{\partial}{\partial\lambda} + \frac{v}{r_s}\frac{\partial}{\partial\varphi} + \omega\frac{\partial}{\partial p}. \tag{11}$$

The pressure-co-ordinate Quasi-Hydrostatic equations are

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{r_s\cos\varphi}\right)(v\sin\varphi - \tilde{w}\cos\varphi) + \frac{1}{r_s\cos\varphi}\frac{\partial\Phi}{\partial\lambda} = F_\lambda \tag{12}$$

$$\frac{Dv}{Dt} + \left(2\Omega + \frac{u}{r_s\cos\varphi}\right)(u\sin\varphi + \frac{v\tilde{w}}{r_s} + \frac{1}{r_s}\frac{\partial\Phi}{\partial\varphi} = F_\varphi \tag{13}$$

$$\frac{RT}{p} + \left(\frac{RT_s}{p}\right)\frac{2\Omega\, ur_s\cos\varphi + u^2 + v^2}{r_s g} + \frac{\partial\Phi}{\partial p} = 0 \tag{14}$$

$$\frac{D\theta}{Dt} = F_\theta \tag{15}$$

$$\frac{1}{r_s\cos\varphi}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial(v\cos\varphi)}{\partial\varphi}\right) + \frac{1}{r_s^2}\frac{\partial(r_s^2\omega)}{\partial p} = 0 \tag{16}$$

For this equation set we also have $\frac{Dq}{Dt} = 0$ for frictionless, adiabatic flow.

Noting the following relationships between derivatives

$$\frac{\partial}{\partial p} = -\frac{RT_s}{gp}\frac{\partial}{\partial r_s} \quad \text{and} \quad w\frac{\partial}{\partial p} = \tilde{w}\frac{\partial}{\partial r_s},$$

we may write down the expressions for the potential vorticity on pressure levels:

$$q|_p = g\left[\frac{1}{r_s^2\cos\varphi}\frac{\partial(r_s v)}{\partial p}\frac{\partial\theta}{\partial\lambda} + \frac{1}{r_s}2\Omega\cos\varphi\frac{\partial\theta}{\partial\varphi} - \frac{1}{r_s^2}\frac{\partial(r_s u)}{\partial p}\frac{\partial\theta}{\partial\varphi}\right.$$

$$\left. -\left(2\Omega\sin\varphi + \frac{1}{r_s\cos\varphi}\left(\frac{\partial v}{\partial\lambda} - \frac{\partial(u\cos\varphi)}{\partial\varphi}\right)\frac{\partial\theta}{\partial\varphi}\right], \qquad (17)$$

and on isentropic surfaces:

$$q|_\theta = \left[\frac{-2\Omega\cos\varphi}{r_s}\frac{\partial r_s}{\partial\varphi} + 2\Omega\sin\varphi + \frac{1}{r_s^2\cos\varphi}\left(\frac{\partial(r_s v)}{\partial\lambda} - \frac{\partial(r_s\cos\varphi)}{\partial\varphi}\right)\right]$$

$$\times\frac{\partial\theta}{\partial p}\left(\left(2\Omega + \frac{u}{r_s\cos\varphi}\right)u\cos\varphi + \frac{v^2}{r_s} - g\right). \qquad (18)$$

For (18), r_s , *u* and *v* must be calculated on θ surfaces, and because the angular derivatives in (18) are taken at constant θ , $r_s(=r_s(p))$, no longer commutes with these operations.

THE CALCULATION OF EQUATION (17)

The Ertel Potential Vorticity on pressure surfaces is calculated in subroutine CALC_PV_P.

The finite difference form of equation (17), with notation as for Unified Model Documentation Paper No.10 unless otherwise stated, is

$$q|_p = g\Bigg[\frac{1}{r_s^2(p)\cos\varphi}\delta_{pa}(r_s(p)v(p))\,\delta_{2\lambda}\theta(p) + \frac{1}{r_{s(p)}}2\Omega\cos\varphi\,\delta_{2\varphi}\theta(p$$

$$-\frac{1}{r_s^2}(p)\,\delta_{pa}(r_s(p)u(p))\,\delta_{2\varphi}\theta(p)$$

$$-\left(2\Omega\sin\varphi + \frac{1}{r_s(p)\cos\varphi}\left(\overline{\delta_\lambda v(p)}^{\varphi} - \overline{\delta_\varphi(u(p)\cos\varphi)}^{\lambda}\right)\right)\mathbb{I}_3[\delta_{2p}\theta](p)\Bigg] \tag{19}$$

where $\delta_{pa}(X) = \dfrac{X(p_-) - X(p_+)}{p_- - p_+}$ (for $X = u$, v and r_s),

$X(p) = I_1[X](p)$ (for n = 1 or 4 (see below)),

$\theta(p) = I_n[\theta](p)$ (for n = 1 or 4 (see below)),

and $I_m[X](p)$ = the m^{th} degree Lagrange interpolation of X in the vertical, valid at a value p.

For the vertical interpolation of theta, a quartic Lagrange polynomial is fitted through five points surrounding the desired pressure level. If no quartic can be fitted due to an unstable or near-unstable model profile, then linear interpolation is used.

The value of $q|_p$ is returned as missing data if no vertical interpolation to the pressure level is possible or terms required for horizontal averaging or differencing fall outside the domain. In the global model the polar value is set to the mean value of the surrounding row.

THE CALCULATION OF EQUATION (18)

The Ertel Potential Vorticity on isentropic surfaces is calculated in subroutine CALC_PV.

The finite difference form of equation (18), with notation as for Unified Model Documentation Paper No.10 unless otherwise stated, is

$$q|_{\theta} = \left[\frac{-2\Omega \cos\varphi}{r_s(\theta)} \delta_{2\varphi} r_s(\theta) \right.$$

qp = ————— δ r (Θ) + 2Ωsinv +
————— δ (r (Θ)v(Θ)) - δ (r (Θ)u(Θ)cosv)
x I [δ Θ](Θ) | 2Ω+————— u(Θ)cosv + ———— - g |(20)

where X(Θ) = I [X](Θ) (for X = p, u, v and r) ,
and I [X](Θ) = the m degree Lagrange interpolation of X in the vertical, valid at a value theta.

The value of qp is returned as missing data if no vertical interpolation to the theta level is possible, or if terms required for horizontal averaging or differencing fall outside the domain. In the global model the polar value is set to the mean value of the surrounding row.

THE_CALCULATION_OF_THETA_ON_POTENTIAL_VORTICITY_SURFACES

The Potential Temperature (theta) on potential vorticity surfaces is calculated using subroutine THETA_PV.

This diagnostic is obtained by first calculating qp and Θp for some specified pressure surfaces by using the method described earlier.

N.B. The diagnostic depends on the pressure surfaces chosen. A different set of vertical levels may lead to a different answer. A default list for finding typical q values in the range 0 - 5 pv units is kept in the User Interface, section B.5.F, and is reproduced here:

950mb, 900mb, 850mb, 800mb, 750mb, 700mb, 650mb, 600mb, 550mb, 500mb, 450mb, 400mb, 350mb, 300mb, 250mb, 200mb, 150mb, 100mb.

The subroutine assumes that the array of pv values chosen is in pv units rather than mks units. Also, if the model domain includes regions from the Southern Hemisphere then the subroutine will automatically search for minus the required pv values in those regions (e.g. if pv=2 is requested, the code will automatically search for pv=-2 in the Southern Hemisphere).

For the vertical interpolation of theta to a pv value, a quartic Lagrange polynomial is fitted through five points surrounding the desired pv surface. If no quartic can be fitted due to an unstable or near-unstable model profile, then linear interpolation is used.

REFERENCES

Cullen, M. J. P., Davies, T. and Mawson, M. H.: Conservative finite difference schemes for a unified forecast/climate model. Unified_Model Documentation_Paper_No.10.

White, A. A. and Bromley, R. A.: Global quasi-hydrostatic models having complete representation of the Coriolis force. Met. Office preprint, March 1990.