UNIFIED MODEL DOCUMENTATION PAPER N0: 14

SOLUTION OF POISSON'S EQUATION ON THE SPHERE USING FOURIER TRANSFORMS.

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Version No 1.

9th October 1992

Model version 2.7

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Modification record		
Document version	Author	Description ....

1. Introduction.

We seek to solve the following equation

$$\nabla^2 Q = R$$

on the surface of a sphere. Given cartesian co-ordinates (x,y) we define spherical polar co-ordinates (λ,ϕ) by $x = a\lambda\cos\phi$, $y = a\phi$, where a is the radius of the sphere. The equation is then,

$$\frac{1}{a^2\cos^2\phi}\frac{\partial^2 Q}{\partial\lambda^2} + \frac{1}{a^2\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial Q}{\partial\phi}\right) = R \qquad (1)$$

Theorem:

There exists a unique solution, up to an arbitrary constant, of equation (1), if and only if, the integral of R over the surface of the sphere is 0.

This condition on R is known as the '*Compatibility Condition*'. From now on we assume that all the functions R we consider possess this property.

We solve equation (1) by decomposing Q and R into fourier series and solving the resulting equations for the coefficients of each wave-number. This approach reduces the 2-dimensional equation into a set of 1-dimensional equations in ϕ. In the code the fourier decomposition is done by application of Fast Fourier Transforms (henceforth FFTs). The 1-dimensional equations in ϕ are discretised using centred finite-differences and the resulting tri-diagonal matrix system is solved directly.

2. Solution Procedure.

Any Riemann integrable, periodic in λ function R can be expressed exactly as a fourier series as follows,

$$R(\lambda,\phi) = r_0(\phi) + is_0(\phi) + \sum_{k=1}^{\infty} r_k(\phi)\cos(k\lambda) + is_k(\phi)\sin(k\lambda)$$

where $i = \sqrt{-1}$. We can write $Q(\lambda,\phi)$ similarly as

$$Q(\lambda,\phi) = a_0(\phi) + ib_0(\phi) + \sum_{k=1}^{\infty} a_k(\phi)\cos(k\lambda) + ib_k(\phi)\sin(k\lambda)$$

Here we consider only real-valued functions of R for which $s_0 = 0$. Substituting these series into equation (1) we arrive at the following set of equations:

Wave-number $k = 0$.

$$\frac{1}{a^2\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial a_o}{\partial\phi}(\phi)\right) = r_0(\phi) \qquad (2)$$

For each wave-number $k > 0$.

$$\frac{-k^2}{a^2\cos\phi}a_k(\phi) + \frac{1}{a^2\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial a_k}{\partial\phi}(\phi)\right) = r_k(\phi) \qquad (3)$$

The same equations with a_k replaced by b_k and r_k replaced by s_k are obtained for the imaginary modes. These equations can now be solved for the coefficients a_k and b_k given the function R . Q is then found by summing the fourier series. The arbitrary constant in the solution to equation (1) comes from equation (2) where it is trivial to observe that the solution a_0 is determined only up to an arbitrary constant. It should also be noted that the solutions to equation (3) are uniquely determined. The usual choice of the arbitrary constant is zero.

3. Numerical Implementation.

We assume that R is given on a numerical grid of n x m points. R is then decomposed into wave-numbers using FFT routines to give components for wave-numbers 0 to $(n/2)$, a total of $(n+2)$ storage locations rather than n . We assume that the solution Q is required at the same points as the right-hand-side R . We discretise equations (2-3) using the centred finite-difference operator

$$\delta_\phi(x_i) = (x_{i-1/2} - x_{i-1/2}) \; / \; \Delta\phi$$

where $i=1$ is the northern most point. This gives rise to a tri-diagonal matrix system which is solved by gaussian elimination. Given that R is real then $s_0 = 0$ and hence $b_0 = 0$. The arbitrary constant in a_0 is determined by setting

$$a_0(\phi) := a_0(\phi) - \overline{a_0(\phi)}^{\phi}$$

where $\overline{a_0\phi}^{\phi}$ is the mean of $a_0(\phi)$ over all values of ϕ . In general the arbitrary constant chosen this way will not be zero but depends on the function $a_0(\phi)$. Depending on the staggering of the numerical grid differing boundary conditions need to be applied to equations (2-3). Two options are available;

3.1.) Variables held at Unified Model Pressure Points.

Subroutine DEL_SQUARED_FFT_P

These pressure points start at the north pole and also include the south pole. At the poles the only non-zero coefficient of R is r_0 . A different boundary condition problem thus exists for this coefficient compared to the others.

For a_0 (equation 2) the boundary condition is :

North Pole : $a_0(\pi/2 + \Delta\phi) = a_0(\pi/2 - \Delta\phi)$

South Pole: $a_0(-\pi/2 - \Delta\phi) = a_0(-\pi/2 + \Delta\phi)$

For a_k and b_k $k>0$ (equation 3) the boundary condition is :

North Pole: $a_k(\pi/2) = 0, \quad b_k(\pi/2) = 0,$

South Pole: $a_k(-\pi/2) = 0, \quad b_k(-\pi/2) = 0,$

3.2.) Variables held at Unified Model Velocity Points.

Subroutine DEL_SQUARED_FFT_U

These pressure points start half a grid-length south of the north pole and stop half a grid-length north of the south pole.

As R has non-zero coefficients for all wave-numbers the same boundary condition is used for equations (2-3). The boundary condition :

at the point next to the North Pole: $a_k(\pi/2 + \Delta\phi/2) = a_k(\pi/2 - \Delta\phi/2)$

at the point next to the South Pole: $a_k(-\pi/2 - \Delta\phi/2) = a_k(-\pi/2 + \Delta\phi/2)$

4. Typical Numerical Errors.

Given a function R which is exactly expressible in terms of wave-numbers 0 to $n/2$, then typically

$$\frac{|| R - \nabla^2 Q ||_2}{|| R ||_2} \approx 10^{-5} \qquad \text{where} \qquad || R ||_2 = \sqrt{\sum_{i=1}^{m*n} R_i^2}$$

If R is not exactly expressible by wave-numbers 0 to $n/2$, for example if R is a square wave periodic in λ, then errors will be introduced by the FFTs. These errors will depend on the exact nature of R and also on the number of waves available to represent R. Code to check whether or not your function R is exactly expressible can be provided by the author upon request.